Int. J. Solids Structures, 1977, Vol. 13, pp. 877–895. Pergamon Press. Printed in Great Britain

FORCED AXISYMMETRIC MOTIONS OF VISCOELASTIC CYLINDRICAL SHELLS

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(*Received* 26 *October* 1976; *revised* 17 *February* 1977)

Abstract—The isothermal response of a viscoelastic cylindrical shell, of finite length, to arbitary axisymmetric surface forces, initial conditions, and boundary conditions is considered within the linear theory of thin shells. The problem is formulated with the effects of shear deformation and rotatory inertia included; the viscoelastic properties are assumed to be isotropic and homogeneous. The response is first found formally in terms of a causal Green's function. It is then shown that when Poisson's ratio is constant, the causal Green's function can be expanded in a series of orthonormal spatial eigenfunctions of an associated elastic shell eigenvalue problem. The resulting solution for the general problem is an eigenfunction series with Laplace transformed time-dependent coefficients. The general solution is applied to predicting the motion of a uniform, simply-supported cylindrical shell, initially quiescent, which is subjected to a step pressure moving with constant velocity. For this example, the relaxation function of the shell material in uniaxial extension is taken to be that of a standard linear solid. The motions predicted by simpler shell models, namely, shells with bending only and without bending, are also considered for comparison. Here, the absolute values of the Fourier coefficients in the shell displacement series go to zero faster than the inverse of the first or second power of positive integers when bending is excluded or included, respectively. Numerical results are presented for a moderately long and relatively thick, nearly elastic, cylindrical shell.

NOTATION

a	radius of shell middle surface
$a_{j,n}$	complex roots of P_n
A, B, C, D	constants
c_p	plate velocity, $c_p = \sqrt{\left(\frac{E_0}{\rho(1-\nu^2)}\right)}$
$C(t), D(t)$	auxiliary relaxation functions defined by eqn (9)
e_{ij}	strain tensor
$E(t)$	relaxation function in uniaxial extension
E_0	$E(0)$
E_∞	$\lim_{t\to\infty} E(t)$
f	dimensionless wave number
$G(t)$	relaxation function in simple shear
G_i^r	components of causal Green's function
$\check{G}_i^a$	components of adjoint Green's function
h	shell wall thickness
$H(..)$	Heaviside step function
κ	shear correction factor
K	constant
l	dimensionless shell length, $l = L/a$
L	shell length
L_{ij}	integro-differential operator components
L_{ij}^{*}	formal adjoint of L_{ij}
$m(x,t)$	external couple per unit area acting about shell middle surface
$M_x(x,t)$	resultant moment per unit length
$N_x(x,t)$	axial stress resultant per unit length
$N_\theta(x,t)$	circumferential stress resultant per unit length
$p(x,t)$	axial pressure on shell middle surface due to external load
P_n	cubic polynomial in s
$q(x,t)$	radial pressure on shell midsurface due to external load
q_0	pressure intensity
s	Laplace transform parameter
t	time
t_1	standard linear solid time constant
T	large positive value of time

$u(x, t)$	axial displacement of shell middle surface
U_i	displacement vector components defined by eqn (15)
v	velocity of moving step pressure
V	dimensionless velocity of step pressure, $V = v/c_p$
$V_x(x, t)$	transverse shear stress resultant per unit length
$\hat{V}$	dimensionless phase velocity
$\hat{V}_0$	dimensionless bar velocity, $\hat{V}_0 = \sqrt{(1-\nu^2)}$
$\hat{V}_{min}$	dimensionless minimum phase velocity
$\hat{V}_P$	dimensionless plate velocity, $\hat{V}_P = 1$
$\hat{V}_s$	dimensionless modified shear velocity, $\hat{V}_s = \gamma$
$w(x, t)$	radial displacement of shell middle surface
W_i	displacement vector components
x	axial coordinate of shell
α	$h/a\sqrt{(12)}$
γ	$\sqrt{(\frac{1}{2}\kappa(1-\nu))}$
$\delta(..)$	Dirac delta function
δ_{ij}	Kronecker delta
$\mathscr{L}$	Laplace transform operator, $\mathscr{L}[f(t)] = \int_0^\infty f(t)\,e^{-st}\,dt$
λ, μ	Lamé constants
$\lambda(t), \mu(t)$	relaxation functions, analogous to λ and μ
ν	Poisson's ratio
$\nu(t)$	viscoelastic Poisson's ratio
ξ	dimensionless axial shell coordinate, $\xi = x/a$
ρ	mass density
σ_{ij}	stress tensor components
τ	nondimensional time, $\tau = tc_p/a$
Φ_i^n	components of nth orthonormal eigenfunction
$\psi(x, t)$	rotation angle of line initially normal to shell middle surface
ω	separation constant, circular frequency
Ω	$\omega a\sqrt{(\rho(1-\nu^2))}$, dimensionless circular frequency
Indices	$i, j, k, r, s = 1, 2, 3.$

A bar over a symbol denotes its Laplace transform. Repeated indices indicate summation, except as noted. Prime denotes differentiation with respect to argument.

1. INTRODUCTION

The motion of uniform linearly elastic shells has been extensively studied; a recent example for a shell of finite length[1] that included the effects of shear deformation and rotatory inertia provided a solution, in the form of the sum of an eigenfunction series and a so-called quasistatic solution that is applicable to a wide class of excitations and responses. Extensions of such results to incorporate viscoelastic material behavior can be accomplished in some cases by the use of the elastic-viscoelastic correspondence principle[2]. However, when the available elastodynamic solution involves an eigenfunction expansion, such as for the cylindrical shell, it is not at all obvious how this analogy should be applied. In such cases, it is usful to construct the entire viscoelastic solution without relying directly upon the elastic results.

The forced axisymmetric motion of finite length viscoelastic cylindrical shells has been studied in [3]. It is shown there that if Poisson's ratio is constant, then a solution can be constructed from the results of three associated analyses involving a quasi-static viscoelastic shell problem, an elastic shell eigenvalue problem and a system of ordinary differential equations. The reasons for assuming a constant Poisson's ratio are here reexamined. With this assumption, the solution for the viscoelastic shell can be obtained entirely in terms of the eigenfunctions of the associated elastic shell eigenvalue problem. This circumvents the separate solution of the quasi-static case which, in some instances, is of the same order of difficulty as the original problem. Such a circumstance arises, for example, in determining the response of a viscoelastic cylindrical shell to a moving axisymmetric pressure pulse which is treated as an application of the derived general results.

The governing relations for a finite length viscoelastic cylindrical shell subject to axisymmetric disturbances are formulated in Section 2. The shell equations include shear deformation and rotatory inertia; it is assumed that the response is isothermal and that the viscoelastic properties, represented in integral form, are isotropic and homogeneous. The solution to the general problem, developed in Section 3, utilizes the method presented in [4] for determining the dynamic response of bounded, three-dimensional viscoelastic bodies; it is first constructed formally in terms of a causal Green's function with the aid of a suitable form of Green's theorem. The task is therby reduced to finding the causal Green's function which, in the present case, constitutes a problem in free vibration. With the assumption of a constant Poisson's ratio, the method of separation of variables leads in a natural way to a causal Green's function expanded in terms of the spatial eigenfunctions associated with an elastic shell. An eigenfunction series solution having Laplace transformed time-dependent coefficients is then obtained for the general shell problem.

In Section 4, the results derived in Section 3 are applied to a viscoelastic cylindrical shell subject to an axisymmetric step pressure travelling with constant velocity. For illustrative purposes, the shell material is assumed to correspond to a nearly elastic standard linear solid. Solutions to the problem given by simpler shell models, that is, shells exhibiting only bending (Love's first approximation) and no bending (membrane theory) are also treated. The convergence of the displacement series is examined for all three shell models. Numerical results are then presented which illustrate the character of the shell motion over a wide range of load speeds and the degree of adequacy of the simpler shell models.

2. STATEMENT OF THE GENERAL PROBLEM

The equations of motion for a finite length viscoelastic cylindrical shell subject to axisymmetric disturbances are presented with the effects of transverse shear deformation and rotatory inertia included to allow for the possibility of the generation of a high frequency response component. The dispersion curves of elastic cylindrical shells incorporating these factors have been found to be in good correspondence with those from an exact three-dimensional model of a hollow elastic cylinder[5]; in particular, the close agreement at the higher frequencies is a direct result of this inclusion. A model which gives a good description of the behavior of elastic cylindrical shells was constructed by Hermann and Mirsky[5] and by Naghdi and Cooper[6] (see also [7]) using somewhat different procedures. The viscoelastic shell equations can also be derived by arguments that parallel either of these elastic derivations; in this study the variational approach used in [6] and [7] will be employed.

With reference to Fig. 1, the dynamic shell equations are obtained by integrating the three-dimensional stress equations of motion across the thickness of the shell after substituting an assumed approximate displacement field. This yields

$$\begin{aligned} \frac{\partial N_x}{\partial x} &= \rho h \frac{\partial^2 u}{\partial t^2} + \frac{\rho h^3}{12a}\frac{\partial^2 \psi}{\partial t^2} + p, \\ \frac{\partial V_x}{\partial x} - \frac{N_\theta}{a} &= \rho h \frac{\partial^2 w}{\partial t^2} + q, \\ \frac{\partial M_x}{\partial x} - V_x &= \frac{\rho h^3}{12}\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{1}{a}\frac{\partial^2 u}{\partial t^2}\right) + m. \end{aligned} \tag{1}$$

The elastic shell stress-displacement relations based on the usual homogeneous, isotropic stress-strain relations were deduced in [6] by means of a variational theorem in elastostatics due to Reissner[8] using the same approximate displacement field referred to above in conjunction with a consistent stress field.

For a homogeneous and isotropic linearly viscoelastic material, initially quiescent and stress free, the isothermal constitutive equations are

$$\sigma_{ij}(t) = \delta_{ij}\int_{0^-}^{t} \lambda(t-\tau)\frac{\partial e_{kk}(\tau)}{\partial \tau}\,d\tau + 2\int_{0^-}^{t} \mu(t-\tau)\frac{\partial e_{ij}(\tau)}{\partial \tau}\,d\tau, \tag{2}$$

where $\lambda(t)$ and $\mu(t)$ are relaxation functions analogous to the Lamé constants in linear elasticity. The corresponding shell stress-displacement relations can be obtained by a generalization of Reissner's variational theorem as developed by Gurtin[9]. An alternate approach[2], followed here, is based on the identical structure of the Laplace transformed equations of a quasi-static viscoelastic boundary value problem, and the equations of an elastostatic boundary problem, if the usual requirements of the elastic-viscoelastic analogy are observed. Reissner's elastostatic variational theorem can then be reinterpreted as a quasi-static

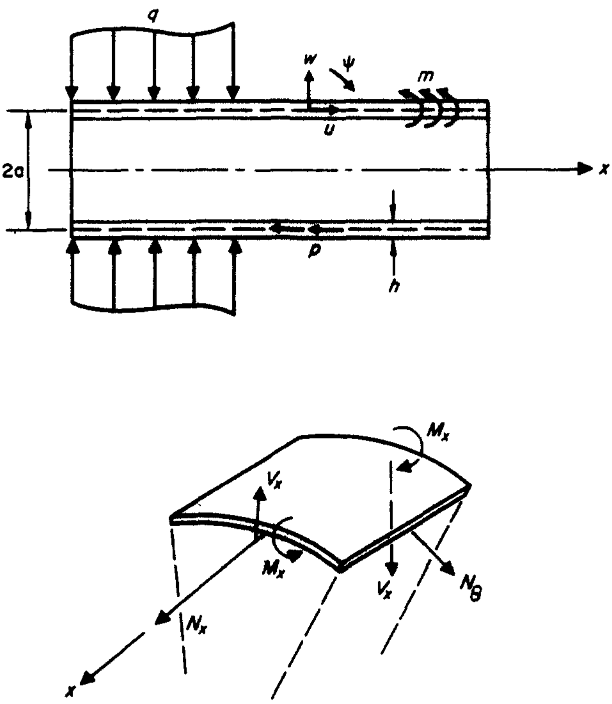

Fig. 1. Schematic view of shell with sign convention.

viscoelastic one involving Laplace transformed field variables. The Euler equations and natural boundary conditions derived from this theorem can thus be regarded as Laplace transformed viscoelastic Euler equations and natural boundary conditions.

If the derivation in [6] is so interpreted, then the desired viscoelastic shell stress-displacement relations follow immediately, since it is only necessary to replace the elastic field variables by their Laplace transforms, and Young's modulus E and Poisson's ratio ν by $s\bar{E}(s)$ and $s\bar{\nu}(s)$, respectively, where s is the transform parameter and the superior bar denotes the transform. The transformed Young's modulus and Poisson's ratio are defined as

$$\bar{E} = \frac{\bar{\mu}(3\bar{\lambda} + 2\bar{\mu})}{\bar{\lambda} + \bar{\mu}}, \quad \text{and} \quad \bar{\nu} = \frac{\bar{\lambda}}{2s(\bar{\lambda} + \bar{\mu})}. \tag{3}$$

It also follows from the viscoelastic reinterpretation of the derivation in [6] that the natural boundary condition for a viscoelastic shell are the same as those for an elastic shell, that is, either the shell stress resultants or the shell displacements must be prescribed at the edges. In summary, the stress-displacement relations for a viscoelastic shell are:

$$\begin{aligned}
\bar{N}_x &= \frac{s\bar{E}h}{1 - s^2\bar{\nu}^2}\left(\frac{\partial \bar{u}}{\partial x} + \frac{s\bar{\nu}\bar{w}}{a} + \frac{h^2}{12a}\frac{\partial \bar{\psi}}{\partial x}\right), \\
\bar{N}_\phi &= \frac{s\bar{E}h}{1 - s^2\bar{\nu}^2}\left(\frac{\bar{w}}{a} + s\bar{\nu}\frac{\partial \bar{u}}{\partial x}\right), \\
\bar{M}_x &= \frac{s\bar{E}h^3}{12(1 - s^2\bar{\nu}^2)}\left(\frac{\partial \bar{\psi}}{\partial x} + \frac{1}{a}\frac{\partial \bar{u}}{\partial x}\right), \\
\bar{V}_x &= s\kappa\bar{G}h\left(\frac{\partial \bar{w}}{\partial x} + \bar{\psi}\right),
\end{aligned} \tag{4}$$

and the boundary conditions are

$$u(x, t) \quad \text{or} \quad N_x(x, t),$$

and

$$w(x, t) \quad \text{or} \quad V_x(x, t), \tag{5}$$

and

$$\psi(x, t) \quad \text{or} \quad M_x(x, t).$$

that must be prescribed on the shell edges of constant x. Above, κ is the shear correction factor which assumes the value 5/6 when the variational derivation is followed, and $\bar{G}$ equals $\bar{\mu}$, the transformed relaxation function in simple shear. It should be noted that terms of order equal to or higher than h^2/a^2 have been neglected in (4).

The boundary value problem of an axisymmetrically excited, viscoelastic cylindrical shell of length L occupying the region $0 \le x \le L$ can now be formulated. A solution for $t > 0$ is sought which satisfies the following governing equations and conditions in the specified domains:

Equations of motion: eqns (1), for $0 < x < L$, $t > 0$;
Stress-displacement relations: eqns (4), for $0 < x < L$;
Initial conditions, for $0 < x < L$:

$$u(x,0) = u_0(x), \quad w(x,0) = w_0(x), \quad \psi(x,0) = \psi_0(x);$$

$$\frac{\partial u}{\partial t}(x,0) = \dot{u}_0(x), \quad \frac{\partial w}{\partial t}(x,0) = \dot{w}_0(x), \quad \frac{\partial \psi}{\partial t}(x,0) = \dot{\phi}_0(x). \tag{6}$$

Boundary conditions (5) must be given at $x = 0$ and $x = L$, for $t > 0$.

It will be assumed henceforth that $N_x(x, t)$, $w(x, t)$ and $M_x(x, t)$ are the boundary conditions specified at the shell edges. In addition, the following two transformed relaxation functions are defined:

$$\bar{C} \equiv \frac{\bar{\bar{E}}}{1 - s^2\bar{\nu}^2}, \quad \bar{D} \equiv s\bar{\nu}\bar{C}. \tag{7}$$

With (7), the transformed stress-displacement relations (4) can be inverted to yield

$$N_x = h\int_{0^-}^{t} C(t-\tau)\left[\frac{\partial^2 u(x,\tau)}{\partial\tau\partial x} + \frac{h^2}{12a}\frac{\partial^2\psi(x,\tau)}{\partial\tau\partial x}\right]\mathrm{d}\tau + \frac{h}{a}\int_{0^-}^{t} D(t-\tau)\frac{\partial w(x,\tau)}{\partial\tau}\mathrm{d}\tau, \tag{8}$$

and the remaining stress resultants can be treated similarly.

Alternate statement of general problem

The general problem can be formulated equivalently by introducing the initial conditions as surface forces, yielding

$$\frac{\partial N_x}{\partial x} + \rho h\left[u_0\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} + \dot{u}_0\delta(t)\right] + \frac{\rho h^3}{12a}\left[\psi_0\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} + \dot{\psi}_0\delta(t)\right] = \rho h\frac{\partial^2 u}{\partial t^2} + \frac{\rho h^3}{12a}\frac{\partial^2\psi}{\partial t^2} + pH(t),$$

$$\frac{\partial V_x}{\partial x} - \frac{N_\theta}{a} + \rho h\left[w_0\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} + \dot{w}_0\delta(t)\right] = \rho h\frac{\partial^2 w}{\partial t^2} + qH(t), \tag{9}$$

$$\frac{\partial M_x}{\partial x} - V_x + \frac{\rho h^3}{12}\left[\left(\psi_0 + \frac{u_0}{a}\right)\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} + \left(\dot{\psi}_0 + \frac{\dot{u}_0}{a}\right)\delta(t)\right] = \frac{\rho h^3}{12}\left(\frac{\partial^2\psi}{\partial t^2} + \frac{1}{a}\frac{\partial^2 u}{\partial t^2}\right) + mH(t),$$

for $0 < x < L$ and $-\infty < t < \infty$, where $\delta(t)$ is the Dirac delta function and $H(t)$ is the Heaviside function;

$$N_x = h\int_{0^-}^{\infty}\left[\frac{\partial u(x,t-s)}{\partial x} + \frac{h^2}{12a}\frac{\partial\psi(x,t-s)}{\partial x}\right]\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s + \frac{h}{a}\int_{0^-}^{\infty} w(x,t-s)\frac{\mathrm{d}D(s)}{\mathrm{d}s}\mathrm{d}s,$$

$$N_\theta = \frac{h}{a}\int_{0^-}^{\infty} w(x,t-s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s + h\int_{0^-}^{\infty}\frac{\partial u(x,t-s)}{\partial x}\frac{\mathrm{d}D(s)}{\mathrm{d}s}\mathrm{d}s, \tag{10}$$

$$M_x = \frac{h^3}{12}\int_{0^-}^{\infty}\left[\frac{\partial\psi(x,t-s)}{\partial x} + \frac{1}{a}\frac{\partial u(x,t-s)}{\partial x}\right]\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s,$$

$$V_x = \kappa h\int_{0^-}^{\infty}\left[\frac{\partial w(x,t-s)}{\partial x} + \psi(x,t-s)\right]\frac{\mathrm{d}G(s)}{\mathrm{d}s}\mathrm{d}s,$$

for $0<x<L$ and $-\infty<t<\infty$, with $u(x,t)=w(x,t)=\psi(x,t)\equiv 0$ for $0<x<L$ and $t<0$, and $N_x(x,t)$, $w(x,t)$ and $M_x(x,t)$ given at $x=0$ and $x=L$ for $t>0$.

To simplify the analysis, the shell displacements u, w, ψ are defined to be the components U_i, $i=1, 2, 3$, of a displacement vector U, such that

$$U_1=u, \quad U_2=w, \quad U_3=\psi. \tag{11}$$

Operators L_{ij}, $i,j=1, 2, 3$ are also defined as

$$\begin{aligned}
L_{11}&=\rho h\frac{\partial^2(\ldots)}{\partial t^2}-h\int_{0^-}^{\infty}\frac{\partial^2(\ldots)}{\partial x^2}(x,t-s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s,\\
L_{12}&=-\frac{h}{A}\int_{0^-}^{\infty}\frac{\partial(\ldots)}{\partial x}(x,t-s)\frac{\mathrm{d}D(s)}{\mathrm{d}s}\mathrm{d}s,\\
L_{13}&=\frac{\rho h^3}{12a}\frac{\partial^2(\ldots)}{\partial t^2}-\frac{h^3}{12a}\int_{0^-}^{\infty}\frac{\partial^2(x..)}{\partial x^2}(x,t-s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s,\\
L_{21}&=-L_{12},\\
L_{22}&=\rho h\frac{\partial^2(\ldots)}{\partial t^2}-\kappa h\int_{0^-}^{\infty}\frac{\partial^2(\ldots)}{\partial x^2}(x,t-s)\frac{\mathrm{d}G(s)}{\mathrm{d}s}\mathrm{d}s+\frac{h}{a}\int_{0^-}^{\infty}(\ldots)(x,t-s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s,\\
L_{23}&=-\kappa h\int_{0^-}^{\infty}\frac{\partial(\ldots)}{\partial x}(x,t-s)\frac{\mathrm{d}G(s)}{\mathrm{d}s}\mathrm{d}s,\\
L_{31}&=L_{13},\\
L_{32}&=-L_{23},\\
L_{33}&=\frac{\rho h^3}{12}\frac{\partial^2(\ldots)}{\partial t^2}-\frac{h^3}{12}\int_{0^-}^{\infty}\frac{\partial^2(\ldots)}{\partial x^2}(x,t-s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s+\kappa h\int_{0^-}^{\infty}(\ldots)(x,t-s)\frac{\mathrm{d}G(s)}{\mathrm{d}s}\mathrm{d}s.
\end{aligned} \tag{12}$$

Let $N_x(U)$ denote N_x evaluated with displacements U_i, $i=1, 2, 3$ in eqn (10), and let the remaining stress resultants be similarly denoted. Then, adopting the indicial notation and indicating summation from 1 to 3 by repeated indices, it follows that

$$\begin{aligned}
L_{1j}U_j&=\rho h\frac{\partial^2 U_1}{\partial t^2}+\frac{\rho h^3}{12a}\frac{\partial^2 U_3}{\partial t^2}-\frac{\partial N_x(U)}{\partial x},\\
L_{2j}U_j&=\rho h\frac{\partial^2 U_2}{\partial t^2}-\frac{\partial V_x(U)}{\partial x}+\frac{N_\phi(U)}{a},\\
L_{3j}U_j&=\frac{\rho h^2}{12}\left(\frac{\partial^2 U_3}{\partial t^2}+\frac{1}{a}\frac{\partial^2 U_1}{\partial t^2}\right)-\frac{\partial M_x(U)}{\partial x}+V_x(U).
\end{aligned} \tag{13}$$

In a manner entirely analogous to that followed in [4], one can define operators L_{ij}^* which are the formal adjoints of L_{ij}. For example,

$$L_{11}^*=\rho h\frac{\partial^2(\ldots)}{\partial t^2}-h\int_{0^-}^{\infty}\frac{\partial^2(\ldots)}{\partial x^2}(x,t+s)\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s. \tag{14}$$

Symbolically, the only difference between the L_{ij} operator and its formal adjoint L_{ij}^* is that the $t-s$ argument in L_{ij} becomes $t+s$ in L_{ij}^*. Next, define stress resultant "adjoints" by

$$\begin{aligned}
N_x^*(U)=h&\int_{0^-}^{\infty}\left[\frac{\partial u(x,t+s)}{\partial x}+\frac{h^2}{12a}\frac{\partial\psi(x,t+s)}{\partial x}\right]\frac{\mathrm{d}C(s)}{\mathrm{d}s}\mathrm{d}s\\
&+\frac{h}{a}\int_{0^-}^{\infty}w(x,t+s)\frac{\mathrm{d}D(s)}{\mathrm{d}s}\mathrm{d}s, \quad \text{etc;}
\end{aligned} \tag{15}$$

the difference between the starred and unstarred quantities is the same as for the L_{ij} and L_{ij}^* operators. Let $N_x^*(W)$ denote N_x^* evaluated with displacement vector components W_i, and denote the remaining stress resultant adjoints similarly. Then, it follows that

$$\begin{aligned} L_{1j}^* W_j &= \rho h \frac{\partial^2 W_1}{\partial t^2} + \frac{\rho h^3}{12a}\frac{\partial^2 W_3}{\partial t^2} - \frac{\partial N_x^*(W)}{\partial x}, \\ L_{2j}^* W_j &= \rho h \frac{\partial^2 W_2}{\partial t^2} - \frac{\partial V_x^*(W)}{\partial x} + \frac{N_\theta^*(W)}{a}, \\ L_{3j}^* W_j &= \frac{\rho h^3}{12}\left(\frac{\partial^2 W_3}{\partial t^2} + \frac{1}{a}\frac{\partial^2 W_1}{\partial t^2}\right) - \frac{\partial M_x^*(W)}{\partial x} + V_x^*(W). \end{aligned} \tag{16}$$

A suitable form of Green's theorem will be needed for the L_{ij} operator as applied to the finite two-dimensional region $0<x<L$, $0<t<T$, where T is a large positive number which, as will be seen later, disappears in the final analysis. For this purpose, assume displacements $U_i(x,t)\equiv 0$ for $t<0$ and $W_i(x,t)\equiv 0$ for $t>T$; then

$$\begin{aligned} \int_0^T \mathrm{d}T \int_0^L \mathrm{d}x\,(W_i L_{ij} U_j - U_i L_{ij}^* W_j) = \int_0^T \mathrm{d}t\,[&U_1 N_x^*(W) - W_1 N_x(U) + U_2 V_x^*(W) - W_2 V_x(U) \\ &+ U_3 M_x^*(W) - W_3 M_x(U)]\big|_0^L \\ &+ h\int_0^L \rho\,\mathrm{d}x\Big[W_1\left(\frac{\partial U_1}{\partial t} + \frac{h^2}{12a}\frac{\partial U_3}{\partial t}\right) - U_1\left(\frac{\partial W_1}{\partial t} + \frac{h^2}{12a}\frac{\partial W_3}{\partial t}\right) \\ &+ W_3\frac{\partial U_2}{\partial t} - U_2\frac{\partial W_2}{\partial t} + \frac{h^2}{12}W_3\left(\frac{\partial U_3}{\partial t} + \frac{1}{a}\frac{\partial U_1}{\partial t}\right) \\ &- \frac{h^2}{12}U_3\left(\frac{\partial W_3}{\partial t} + \frac{1}{a}\frac{\partial W_1}{\partial t}\right)\Big]\Big|_0^T. \end{aligned} \tag{17}$$

3. SOLUTION TO THE GENERAL PROBLEM

Causal Green's function

Define the causal Green's function, $G_i^r(x,t|x_0,t_0)$ for the region $0<x<L$, as the solution to the following boundary value problem:

$$L_{ij}G_j^r(x,t|x_0,t_0) = \delta_i^r(x-x_0)\delta(t-t_0), \tag{18}$$

for $0<x, x_0<L$ and $-\infty<t, t_0<\infty$;

$$G_i^r(x,t|x_0,t_0)\equiv 0, \tag{19}$$

for $t<t_0$ and $0<x, x_0<L$;

$$N_x(G^r)=0, \qquad G_2^r=0, \qquad M_x(G^r)=0 \tag{20}$$

at $x=0$ and $x=L$. The last set of relations constitutes the homogeneous form of the boundary conditions specified for the general problem.

Above, the index r takes on the values 1, 2 and 3, $\delta_i^r(x)=0$ when $i\neq r$, and $\delta_i^r(x)=\delta(x)$ when $i=r$. The function $\delta(x)$ is the one-dimensional Dirac delta function. Physically, $G_i^r(x,t|x_0,t_0)$ is the ith displacement component, at x and t, due to a unit concentrated surface force applied in the direction of the rth displacement component, at x_0 and t_0.

It will be shown that the general problem defined by eqns (9) and (10) can be solved in terms of G^r. For this purpose, Green's theorem in the form of eqn (17) will be used with U as the solution of these equations, and W an appropriate Green's function. Since the Green's function for the operator L_{ij}^* will be needed and since this function will have to vanish for $t=T$, one is led to consider the adjoint problem

$$L_{ij}^{*}\,\tilde{G}_j^s(x, t|x_0, t_0) = \delta_i^s(x - x_0)\delta(t - t_0), \tag{21}$$

for $0 < x,\ x_0 < L$ and $-\infty < t,\ t_0 < \infty$;

$$\tilde{G}_i^s(x, t|x_0, t_0) = 0, \tag{22}$$

for $t > t_0$ and $0 < x,\ x_0 < L$;

$$N_x^*(\tilde{G}^s) = 0, \qquad \tilde{G}_2^s = 0, \qquad M_x^*(\tilde{G}^s) = 0, \tag{23}$$

at $x = 0$ and $x = L$. Equations (23) are the adjoint boundary conditions of the general problem.

By letting $W_i = \tilde{G}_i^s(x, t|x_0, t_0)$ and $U_i = G_i^r(x, t|x_2, t_2)$ in eqn (17) and replacing $(0, T)$ by (T_1, T_2), where $T_1 < t_0$, $t_2 < T_2$, one can show that

$$\tilde{G}_r^s(x_2, t_2|x_0, t_0) = G_s^r(x_0, t_0|x_2, t_2). \tag{24}$$

Dropping the "two" subscripts in (24) gives

$$\tilde{G}_r^s(x, t|x_0, t_0) = G_s^r(x_0, t_0|x, t). \tag{25}$$

Another useful result follows from eqns (18)–(20) by making the change of variables $t' = t - t_0$, namely

$$G_i^r(x, t|x_0, t_0) = G_i^r(x, t - t_0|x_0, 0). \tag{26}$$

General solution

Assume $T > t_0$, let U_i be the solution of eqns (9) and (10), $W_i = \tilde{G}_i^r(x, t|x_0, t_0) = G_r^i(x_0, t_0|x, t)$ and apply eqn (17). Then, by using the fact that $\partial G_r^i/\partial t_0 = -\partial G_r^i/\partial t$ and by making several changes of variables as described in more detail in [4], one obtains

$$\begin{aligned}
U_r(x, t) = &-\int_0^t d\tau \int_0^L d\xi [p(\xi, \tau)G_r^1(x, t - \tau|\xi, 0) + q(\xi, \tau)G_r^2(x, t - \tau|\xi, 0) \\
&+ m(\xi, \tau)G_r^3(x, t - \tau|\xi, 0)] + \int_0^t d\tau \Big\{ N_x(\xi, \tau)G_r^1(x, t - \tau|\xi, 0) \\
&- \kappa h w(\xi, \tau)\int_{0^-}^{t-\tau}\left[\frac{\partial G_r^2(x, t - \tau|\xi, s)}{\partial \xi} + G_r^3(x, t - \tau|\xi, s)\right]\frac{dG(s)}{ds}\,ds \\
&+ M_x(\xi, \tau)G_r^3(x, t - \tau|\xi, 0)\Big\}\Big|_{\xi=0}^{\xi=L} + h\int_0^L d\xi \rho(\xi)\Big\{\left[\dot{u}_0(\xi) + \frac{h^2}{12a}\dot{\psi}_0(\xi)\right]G_r^1(x, t|\xi, 0) \\
&+ \left[u_0(\xi) + \frac{h^2}{12a}\psi_0(\xi)\right]\frac{dG_r^1(x, t|\xi, 0)}{dt} + \dot{w}_0(\xi)G_r^2(x, t|\xi, 0) + w_0(\xi)\frac{dG_r^2(x, t|\xi, 0)}{dt} \\
&+ \frac{h^2}{12}\left[\dot{\psi}_0(\xi) + \frac{1}{a}\dot{u}_0(\xi)\right]G_r^3(x, t|\xi, 0) + \frac{h^2}{12}\left[\psi_0(\xi) + \frac{1}{a}u_0(\xi)\right]\frac{dG_r^3(x, t|\xi, 0)}{dt}\Big\}.
\end{aligned} \tag{27}$$

Calculating the causal Green's function

Consider the "free vibration" problem,

$$\begin{aligned}
\frac{\partial N_x}{\partial x} &= \rho h\frac{\partial^2 u}{\partial t^2} + \frac{\rho h^3}{12a}\frac{\partial^2 \psi}{\partial t^2}, \\
\frac{\partial V_x}{\partial x} - \frac{N_\theta}{a} &= \rho h\frac{\partial^2 w}{\partial t^2}, \\
\frac{\partial M_x}{\partial x} - V_x &= \frac{\rho h^3}{12}\left(\frac{\partial^2 \psi}{\partial t^2} + \frac{1}{a}\frac{\partial^2 u}{\partial t^2}\right),
\end{aligned} \tag{28}$$

for $0<x<L$ and $t>0$; N_x, N_θ, M_x and V_x are given by expressions such as (8) with initial conditions

$$u(x,0)=0, \qquad w(x,0)=0, \qquad \psi(x,0)=0;$$
$$\frac{\partial u}{\partial t}(x,0)=\dot{u}_0(x), \qquad \frac{\partial w}{\partial t}(x,0)=\dot{w}_0(x), \qquad \frac{\partial \psi}{\partial t}(x,0)=\dot{\psi}_0(x) \tag{29}$$

for $0<x<L$, and boundary conditions

$$N_x(x,t)=0, \qquad w(x,t)=0, \qquad M_x(x,t)=0 \tag{30}$$

at $x=0$ and $x=L$, for $t>0$.

The solutions of the differential equations will be sought in the form of products of functions of space and time. These separable solutions will be required to satisfy the boundary conditions but not necessarily all the initial conditions. By superposition, all the initial conditions will be satisfied later as well. Let

$$\begin{Bmatrix} u(x,t) \\ w(x,t) \\ \psi(x,t) \end{Bmatrix} = \begin{Bmatrix} X_1(x) \\ X_2(x) \\ X_3(x) \end{Bmatrix} T(t) \tag{31}$$

be a separable solution satisfying the boundary conditions; upon substituting eqn (31) in the differential equations, one finds that the equations can be separated if

$$\lambda(t)+\mu(t)=K\mu(t), \tag{32}$$

where K is a constant. This relation implies that Poisson's ratio, $\nu(t)=\nu$ is a constant. With this hypothesis, one obtains through the usual separation of variables argument that

$$\begin{aligned} &\frac{1}{1-\nu^2}\left(X_1''+\frac{\nu}{a}X_2'+\frac{h^2}{12a}X_3''\right)=-\rho\omega^2\left(X_1+\frac{h^2}{12a}X_3\right), \\ &\frac{\kappa}{2(1+\nu)}(X_2''+X_3')-\frac{1}{1-\nu^2}\left(\frac{X_2}{a^2}+\frac{\nu}{a}X_1'\right)=-\rho\omega^2X_2, \\ &\frac{h^2}{12(1-\nu^2)}\left(X_3''+\frac{1}{a}X_1''\right)-\frac{\kappa}{2(1+\nu)}(X_2'+X_3)=-\frac{\rho\omega^2h^2}{12}\left(X_3+\frac{1}{a}X_1\right), \end{aligned} \tag{33}$$

where prime denotes differentiation. Further, the boundary conditions become

$$X_1'(0)=X_1'(L)=0, \qquad X_2(0)=X_2(L)=0, \qquad X_3'(0)=X_3'(L)=0 \tag{34}$$

and also

$$\int_{0^-}^{t} E(t-\tau)\frac{dT}{d\tau}d\tau=-\frac{1}{\omega^2}\frac{d^2T}{dt^2}, \tag{35}$$

where ω^2 is a separation constant. The eigenvalue problem defined by eqns (33) and (34) is precisely that for an elastic cylindrical shell having a Poisson's ratio ν. Hence, one can conclude when the shell length is finite that there is a denumerable sequence of distinct and positive eigenvalues, or natural frequencies,

$$\omega_1^2<\omega_2^2<\cdots<\omega_n^2<\cdots$$

which are listed in increasing order. The corresponding eigenfunctions

$$\Phi_i^1(x), \qquad \Phi_i^2(x),\ldots, \qquad \Phi_i^n(x),\ldots$$

can be chosen to form a complete orthonormal set satisfying

$$\int_0^L h\rho\left(\Phi_1{}^m\Phi_1{}^n+\frac{h^2}{12}\Phi_3{}^m\Phi_3{}^n+\Phi_2{}^m\Phi_2{}^n+\frac{h^2}{12a}\Phi_1{}^m\Phi_3{}^n+\frac{h^2}{12a}\Phi_3{}^m\Phi_1{}^n\right)\mathrm{d}x=\delta_{mn}. \tag{36}$$

By taking the Laplace transform of eqn (35), one obtains for each ω_n

$$s\bar{E}\bar{T}_n=-\omega_{(n)}^{-2}[s^2\bar{T}_{(n)}-sT_{(n)}(0)-T'_{(n)}(0)]. \tag{37}$$

Repeated indices that are not to be summed are placed with parentheses. Setting $T_n(0)=0$, then $\Phi_i{}^{(n)}T_{(n)}$ satisfies the differential equations, the boundary conditions and the first three of the initial conditions (29). Leaving aside any questions regarding convergenc and term-by-term differentiability, the same is true for

$$\sum_{n=1}^{\infty}\Phi_i{}^n\mathscr{L}^{-1}\left[\frac{T'_n(0)}{s\bar{E}+s^2\omega_n{}^{-2}}\right],$$

where the $T'_n(0)$ are constants which are to be adjusted so as to satisfy the last three of the initial conditions (29),

$$\begin{Bmatrix}\dot{u}_0(x)\\ \dot{w}_0(x)\\ \dot{\psi}_0(x)\end{Bmatrix}=\sum_{n=1}^{\infty}\left\{\lim_{s\to\infty}\left[\frac{s^2T'_n(0)}{s\bar{E}+s^2\omega_n{}^{-2}}\right]\right\}\begin{Bmatrix}\Phi_1{}^n\\ \Phi_2{}^n\\ \Phi_3{}^n\end{Bmatrix},$$

$$=\sum_{n=1}^{\infty}\omega_n{}^2T'_n(0)\begin{Bmatrix}\Phi_1{}^n\\ \Phi_2{}^n\\ \Phi_3{}^n\end{Bmatrix}. \tag{38}$$

This equation will hold if and only if

$$\omega_{(n)}^2T'_{(n)}(0)=\int_0^L h\rho\left(\dot{u}_0\Phi_1{}^n+\frac{h^2}{12}\dot{\psi}_0\Phi_3{}^n+\dot{w}_0\Phi_2{}^n+\frac{h^2}{12a}\dot{u}_0\Phi_3{}^n+\frac{h^2}{12a}\dot{\psi}_0\Phi_1{}^n\right)\mathrm{d}x. \tag{39}$$

Thus, the formal solution of eqns (28)–(30) is

$$\begin{Bmatrix}u(x,t)\\ w(x,t)\\ \psi(x,t)\end{Bmatrix}=\sum_{n=1}^{\infty}\mathscr{L}^{-1}\left[\frac{\int_0^L h\rho\left(\dot{u}_0\Phi_1{}^n+\frac{h^2}{12}\dot{\psi}_0\Phi_3{}^n+\dot{w}_0\Phi_2{}^n+\frac{h^2}{12a}\dot{u}_0\Phi_3{}^n+\frac{h^2}{12a}\dot{\psi}_0\Phi_1{}^n\right)\mathrm{d}\xi}{s\omega_n{}^2\bar{E}+s^2}\right]\begin{Bmatrix}\Phi_1{}^n(x)\\ \Phi_2{}^n(x)\\ \Phi_3{}^n(x)\end{Bmatrix}. \tag{40}$$

For the causal Green's function $G_i{}^r$ $(x,t|x_0,0)$, the differential equations, initial and boundary conditions are identical to those of the "free vibration" problem above, except that for the Green's function $G_1{}^r$, the initial velocities are

$$\dot{u}_0(x)=\frac{a\delta(x-x_0)}{\rho h\left(a-\frac{h^2}{12a}\right)},\qquad \dot{w}_0(x)=0,\qquad \dot{\psi}_0(x)=\frac{\delta(x-x_0)}{\rho h\left(\frac{h^2}{12a}-a\right)}; \tag{41}$$

for $G_2{}^r$, the initial velocities are

$$\dot{u}_0(x)=0,\qquad \dot{w}_0(x)=\frac{\delta(x-x_0)}{\rho h},\qquad \dot{\psi}_0(x)=0, \tag{42}$$

and for $G_3{}^r$, the initial velocities are

lth component of $\mathbf{Y}_r$ by employing the mth approximations of the already computed q componts ($q < l$) of $\mathbf{Y}_r$. Thus

$$\mathbf{Y}_r^{(m)} = \mathbf{L}_r Y_r^{(m)} + \mathbf{U}_r \mathbf{Y}_r^{(m-1)} + \mathbf{F}_r \quad m = 1, 2, \ldots \tag{15}$$

where $\mathbf{L}_r$ and $\mathbf{U}_r$ are the lower and upper triangular matrices of $\mathbf{E}_r$, respectively. For $\mathbf{Y}_r^{(0)}$ we employ the values of $\mathbf{Y}_r$ in the previous time step $n-1$.

In order to show that the iterative process (15) converges, we employ theorem (3.4) in Varga[10], which asserts that if the matrices $\mathbf{I} - \mathbf{E}_r$ are irreducibly diagonally dominant then the iterative procedure (15) for the equations $(\mathbf{I} - \mathbf{E}_r)\,\mathbf{Y}_r = \mathbf{F}_r$ are convergent for any initial vectors $\mathbf{Y}_r^{(0)}$.

The matrices $\mathbf{I} - \mathbf{E}_r$ are indeed irreducible since it can be verified by examining the location of the elements (13) in $\mathbf{E}_r$ that their directed graphs[10] are strongly connected. This property expresses the fact that in each one of the system of matrix eqns (11), the equations are coupled and it is not possible to reduce any system to the solution of a lower order matrix equation.

It remains to show that the matrices $\mathbf{I} - \mathbf{E}_r$ are diagonally dominant, i.e.

$$\sum_m |(E_r)_{l,m}| \leq 1 \quad l = 1, 2, \ldots \tag{16}$$

with strict inequality for at least one l. Referring to the typical rows of $\mathbf{E}_r$ given by (13), we obtain the following inequalities

$$\epsilon = \Delta y/2\Delta x \leq 1 \quad \text{for} \quad \Delta y \leq 2\Delta x,$$
$$2\epsilon = \Delta y/\Delta x \leq 1 \quad \text{for} \quad \Delta y \leq \Delta x,$$
$$2\epsilon|\delta| = (\Delta y/\Delta x)|\lambda/(\lambda + 2\mu)| = (\Delta y/\Delta x)\lambda/(\lambda + 2\mu) < 1. \tag{17}$$

In the last inequality, we have utilized the inequalities $\lambda + 2\mu/3 > 0$, $\mu > 0$ for the positive definiteness of the strain energy of an isotropic material, and also the relation $\lambda > 0$ for real materials. Consequently, the matrices $\mathbf{I} - \mathbf{E}_r$ are diagonally dominant for $\Delta y \leq \Delta x$ with strict inequality in (16) for at least one l. Hence the iterative procedure (15) is convergent.

Having computed all the displacements at the boundary $y = 0$ of the half-space, we can calculate the stresses within the assumed contact region and verify that the previous two requirements for the dynamic contact problem are satisfied. If the answer is affirmative, we deduce that the correct solution at time $t = n\Delta t$ has been obtained so that we can proceed to the next time step $t = (n+1)\Delta t$. In the case of a negative answer, we modify the assumed contact point i_0 by passing to a neighboring point and repeat the process by imposing again the boundary conditions (9), (10) with the new value of i_0 and solving the resulting equations for the displacements on the boundary. This iterative process is continued until all the requirements as well as the boundary conditions are satisfied simultaneously yielding the correct contact region. The boundary conditions (3), (4) for a perfect adhesion are treated similarly.

APPLICATIONS

In the following we apply the proposed method of solution to the problem of indentation of a half-space by a wedge-shaped punch and parabolic punch. In some situations analytical solutions are known which can be employed in order to assess the accuracy and reliability of the numerical method. All the results presented in this paper were obtained with the spatial increments $\Delta x = \Delta y = d/50$, where d is a reference measure of length and the time increment $c_1\Delta t/d = 0.01$.

(1) *Smooth indentation by a wedge at a uniform speed*

Consider a rigid wedge-shaped die which indent the half-space at a given constant speed V. It is assumed that the indentation is smooth so that the boundary conditions are given by (2), (3) with (5) and $p(t) = Vt$.

By employing the self-similar method of solution, Robinson and Thompson[2], obtained an

Forcing functions:

$$p(x,t)=m(x,t)=0, \quad \text{for} \quad 0<x<L \quad \text{and} \quad t>0; \tag{48}$$

$$q(x,t)=q_0H(vt-x), \quad \text{for} \quad 0<x<L \quad \text{and} \quad 0<t<L/v; \tag{49}$$

$$q(x,t)=q_0, \quad \text{for} \quad 0<x<L \quad \text{and} \quad L/v\le t. \tag{50}$$

Substitution of conditions (46)–(50) into (45) yields the Laplace transformed solution

$$\bar{U}_r(x,s)=-q_0\sum_{m=1}^{\infty}\frac{\int_0^L e^{-\xi s/v}\Phi_2^m(\xi)\,d\xi}{s(s\omega_m^2\bar{E}+s^2)}\Phi_r^m(x). \tag{51}$$

In order to invert (51), it is necessary to specify $\bar{E}(s)$. For this purpose, it is assumed that

$$E(t)=E_\infty+(E_0-E_\infty)\,e^{-t/t_1}, \tag{52}$$

where E_∞, E_0 and t_1 are constants. This representation of the relaxation function is known as a standard linear solid. The s multiplied Laplace transform of (52) is

$$s\bar{E}(s)=\frac{\frac{E_\infty}{t_1}+E_0s}{s+\frac{1}{t_1}}. \tag{53}$$

Let the cubic polynomial $P_n(s)$ be defined by

$$P_n(s)=\omega_n^2\left(\frac{E_\infty}{t_1}+E_0s\right)+s^2\left(s+\frac{1}{t_1}\right), \tag{54}$$

and write it in factored form as

$$P_n(s)=\prod_{j=1}^{3}(s-a_{j,n}) \tag{55}$$

where $a_{j,n}$ are the complex roots of (54). Using (55), the solution can be inverted to yield

$$U_r(x,t)=-q_0\sum_{n=1}^{\infty}\sum_{k=1}^{3}\frac{a_{k,n}+\frac{1}{t_1}}{\lim\limits_{s\to a_{k,n}}\left[\prod_{j=1}^{3}(s-a_{j,n})\Big/(s-a_{k,n})\right]}$$
$$\times\left[\int_0^L\Phi_2^n(\xi)\int_0^t H(v\tau-\xi)\,e^{a_{k,n}(t-\tau)}\,d\tau\,d\xi\right]\Phi_r^n(x). \tag{56}$$

Before numerical results can be obtained from (56) it is necessary to solve the eigenvalue problem defined by eqns (33) and (34). To this end, assume as prospective solutions of (33) the functions

$$X_1=A\cos m\pi x/L, \quad X_2=B\sin m\pi x/L, \quad X_3=C\cos m\pi x/L, \quad m=0,1,2,\ldots, \tag{57}$$

where A, B, C are constants. It will be observed that the assumed solutions satisfy the boundary conditions (34). Substitution of eqns (57) into (33) gives three linear homogeneous equations in the three unknown constants A, B and C. For a nontrivial solution of these equations to exist, the determinant of the matrix formed by the coefficients of A, B and C must vanish. This leads to the characteristic equation

$$(1-\alpha^2)\Omega^6-\left[\frac{m^2\pi^2}{l^2}(2+\gamma^2)(1-\alpha^2)+(1-\alpha^2)+\frac{\gamma^2}{\alpha^2}\right]\Omega^4$$
$$+\left\{\frac{m^4\pi^4}{l^4}(1+2\gamma^2)(1-\alpha^2)+\frac{m^2\pi^2}{l^2}\left[2(1-\alpha^2)+\frac{\gamma^2}{\alpha^2}(1-\alpha^2\nu)-\nu(\nu+\gamma^2)\right]\right.$$
$$\left.+\frac{\gamma^2}{\alpha^2}\right\}\Omega^2-\left\{\frac{m^6\pi^6}{l^6}\gamma^2(1-\alpha^2)+\frac{m^4\pi^4}{l^4}[1-\alpha^2-\nu(\nu+2\gamma^2)]+\frac{m^2\pi^2\gamma^2}{l^2\alpha^2}(1-\nu^2)\right\}=0, \quad (58)$$

where the following nondimensional parameters have been introduced:

$$\alpha^2=\frac{h^2}{12a^2},\qquad l=\frac{L}{a},\qquad \gamma^2=\frac{\kappa}{2}(1-\nu),\qquad \Omega=\omega a\sqrt{(\rho(1-\nu^2))}. \quad (59)$$

For each value of m, (58) gives three distinct positive roots (eigenvalues) for Ω^2. The eigenvalues corresponding to all $m \geq 1$ can be arranged in an increasing sequence and labelled with the single index n so that

$$\Omega_1^2<\Omega_2^2<\cdots<\Omega_n^2<\cdots. \quad (60)$$

The eigenvalues given by (58) for $m=0$ are left out in (60) because the corresponding eigenfunctions have zero radial displacement components and so do not contribute to the series solution, eqn (56). It should be noted that the ordering of the eigenvalues in (60) makes the harmonic number m a function of the index n.

For every n, the three linear homogeneous equations in A, B and C provide the three constants

$$B_n=1,$$
$$C_n=-\frac{m\pi l(\alpha^2\nu+\gamma^2)}{a\alpha^2(\alpha^2-1)(\Omega_n^2l^2-m^2\pi^2)+\gamma^2l^2}, \quad (61)$$
$$A_n=-\frac{m\pi\nu l}{\Omega_n^2l^2-m^2\pi^2}-\alpha^2aC_n,$$

where B_n was arbitrarily set equal to one. The above constants in conjunction with assumed solutions (57) yield the following orthonormal eigenfunctions satisfying (36):

$$\Phi_1^n=a_n\cos\frac{m\pi x}{L},\qquad \Phi_2^n=b_n\sin\frac{m\pi x}{L},\qquad \Phi_3^n=c_n\cos\frac{m\pi x}{L} \quad (62)$$

where

$$a_n=A_n/\sqrt{D_n},\qquad b_n=B_n/\sqrt{D_n},\qquad c_n=C_n/\sqrt{D_n}\ .$$

and

$$D_n=\frac{\rho hL}{2}\left(A_n^2+\frac{h^2}{12}C_n^2+B_n^2+\frac{h^2}{6a}A_nC_n\right).$$

The step remaining is the evaluation of the double integral in (56). For $0\leq t<L/v$, the double integral gives

$$\int_0^L\Phi_2^n(\xi)\int_0^t H(v\tau-\xi)\,e^{a_{k,n}(t-\tau)}\,d\tau\,d\xi=-\left(\frac{b_n}{a_{k,n}}\right)\left[\frac{1}{1+\left(\frac{m\pi v}{La_{k,n}}\right)^2}\right]\left\{\frac{L}{m\pi}\left(1-\cos\frac{m\pi vt}{L}\right)\right.$$
$$\left.+\frac{v}{a_{k,n}}\sin\frac{m\pi vt}{L}+\frac{m\pi v^2}{La_{k,n}^2}(1-e^{a_{k,n}t})\right\}, \quad (63)$$

and for $t \geq L/v$,

$$\int_0^L \Phi_2{}^n(\xi)\int_0^t H(v\tau-\xi)\,e^{a_{k,n}(t-\tau)}\,d\tau\,d\xi = -\left(\frac{b_n}{a_{k,n}}\right)\left[\frac{1}{1+\left(\frac{m\pi v}{La_{k,n}}\right)^2}\right]\left\{\frac{L}{m\pi}[1-(-1)^m] + \frac{m\pi v^2}{La_{k,n}^2}(1-e^{a_{k,n}t}) + \frac{m\pi v^2}{La_{k,n}^2}(-1)^m[e^{a_{k,n}(t-(L/v))}-1]\right\}, \quad (64)$$

where m, as will be recalled, is a function of n. Thus, the complete solution to the viscoelastic shell problem posed above is given by eqn (56), in conjunction with (63) and (64).

It is now an easy matter to formulate the solutions to the same problem employing simpler shell theories. All that is required is to find the appropriate eigenvalues and eigenfunctions to be used in expressions such as (54) and (56). For instance, if shear deformation and rotatory inertia are neglected in (1) and (8) then the shell equations correspond to the equations commonly referred to as Love's first approximation, the viscoelastic form of which are

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} = \rho h\frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial V_x}{\partial x} - \frac{N_\theta}{a} = \rho h\frac{\partial^2 w}{\partial t^2} + q,\\
&\frac{\partial M_x}{\partial x} - V_x = 0,
\end{aligned} \quad (65)$$

with

$$\begin{aligned}
N_x &= h\int_{0^-}^t C(t-\tau)\frac{\partial^2 u(x,\tau)}{\partial\tau\partial x}\,d\tau + \frac{h}{a}\int_{0^-}^t D(t-\tau)\frac{\partial w(x,\tau)}{\partial\tau}\,d\tau,\\
N_\theta &= \frac{h}{a}\int_{0^-}^t C(t-\tau)\frac{\partial w(x,\tau)}{\partial\tau}\,d\tau + h\int_{0^-}^t D(t-\tau)\frac{\partial^2 u(x,\tau)}{\partial\tau\partial x}\,d\tau,\\
M_x &= -\frac{h^3}{12}\int_{0^-}^t C(t-\tau)\frac{\partial^3 w}{\partial\tau\partial x^2}\,d\tau,
\end{aligned} \quad (66)$$

and with the boundary conditions:

$$\begin{aligned}
w(0,t) &= N_x(0,t) = M_x(0,t) = 0,\\
w(L,t) &= N_x(L,t) = M_x(L,t) = 0.
\end{aligned} \quad (67)$$

The separation of variables argument yields the characteristic equation

$$\Omega^4 - \Omega^2\left(\frac{m^4\pi^4\alpha^2}{l^4} + \frac{m^2\pi^2}{l^2} + 1\right) + \frac{m^2\pi^2}{l^2}\left(\frac{m^4\pi^4\alpha^2}{l^4} + 1 - \nu^2\right) = 0, \quad (68)$$

and the orthonormal eigenfunctions,

$$\begin{aligned}
\Phi_1{}^n &= a_n\cos\frac{m\pi x}{L},\\
\Phi_2{}^n &= b_n\sin\frac{m\pi x}{L},
\end{aligned} \quad (69)$$

where

$$a_n = A_n/\sqrt{D_n}, \qquad b_n = B_n/\sqrt{D_n},$$

$$B_n = 1, \qquad A_n = \frac{m\pi\nu l}{m^2\pi^2 - \Omega_n{}^2 l^2},$$

and

$$D_n = \frac{\rho h L}{2}(A_n^2 + B_n^2).$$

If bending is also neglected, then the shell equations correspond to the membrane equations, the viscoelastic form of which are

$$\frac{\partial N_x}{\partial x} = \rho h \frac{\partial^2 u}{\partial t^2},$$
$$-\frac{N_\theta}{a} = \rho h \frac{\partial^2 w}{\partial t^2} + q, \tag{70}$$

with N_x and N_θ identical to the expressions in (66); the boundary conditions are

$$N_x(0, t) = N_x(L, t) = 0. \tag{71}$$

The separation of variables argument leads to the characteristic equation

$$\Omega^4 - \Omega^2\left(1 + \frac{m^2\pi^2}{l^2}\right) + \frac{m^2\pi^2}{l^2}(1-\nu^2) = 0 \tag{72}$$

and the formulae for the orthonormal eigenfunctions are the same as eqns (69).

Calculated results in dimensionless form are presented that show the character of the shell motion over a wide range of load speeds and the extent of agreement of the various shell models. The governing equations can be put in dimensionless form if the variables having dimensions of length, namely, u, w and x, are divided by the shell radius a, time t is divided by radius over plate velocity a/c_p, where $c_p \equiv \sqrt{(E_0/[\rho(1-\nu^2)])}$, and the stress resultants such as N_x are divided by $E_0h/(1-\nu^2)$; further, let

$$\xi \equiv \frac{x}{a}, \qquad \tau \equiv \frac{c_p t}{a}, \qquad V \equiv \frac{v}{c_p}. \tag{73}$$

The results are based on a specific shell configuration with properties $\nu = 0.3$, $l \equiv L/a = 10$, $h/a = 0.1$, $E_\infty/E_0 = 0.7$, $c_pt_1/a = 100$ and subject to a dimensionless pressure, $q_0a(1-\nu^2)/(E_0h) = -0.00091$.

The radial motions predicted at $\xi = 5.2$ by the three shell models are shown in Figs. 2 and 3. The significance of the load speeds is related to the velocity of propagation of harmonic waves in infinitely long elastic shells with a Young's modulus E_0, but otherwise the same physical and geometric properties as the viscoelastic tube. The normalized velocity of propagation of such harmonic waves can be obtained for the most exact shell model from (58) and for the simpler models from (68) and (72) upon substituting

$$\frac{m\pi}{l} = \hat{f}, \qquad \Omega = \hat{V}\hat{f} \tag{74}$$

where dimensionless phase velocity $\hat{V}$ and wave number $\hat{f}$ correspond to the dimensionless harmonic wave phase, $\hat{f}(\hat{V}\tau - \xi)$. With the assumed shell properties, one obtains the dispersion curves shown in Fig. 4. The most exact shell model yields three modes of propagating harmonic waves for a fixed wave number. The dispersion curves for two of the three modes begin at an infinitely large phase velocity, when the wave number is zero, and then drop asymptotically to $\hat{V}_p = 1$, the dimsionless plate velocity, as the wave number approaches infinity. The dispersion curve for the remaining mode starts at $\hat{V}_0 = \sqrt{(1-\nu^2)}$, the dimensionless bar velocity, and with increasing wave number, it drops to a minimum phase velocity $\hat{V}_{min} \simeq \sqrt{(2\alpha(1-\nu^2)^{1/2})}$, and then rises asymptotically to the modified shear velocity, $\hat{V}_s = \gamma = \sqrt{((\kappa(1-\nu)/2)}$ as the wave number goes to infinity. Load speeds equal to $\hat{V}_{min}$ and $\hat{V}_0$ are of particular interest for the most exact

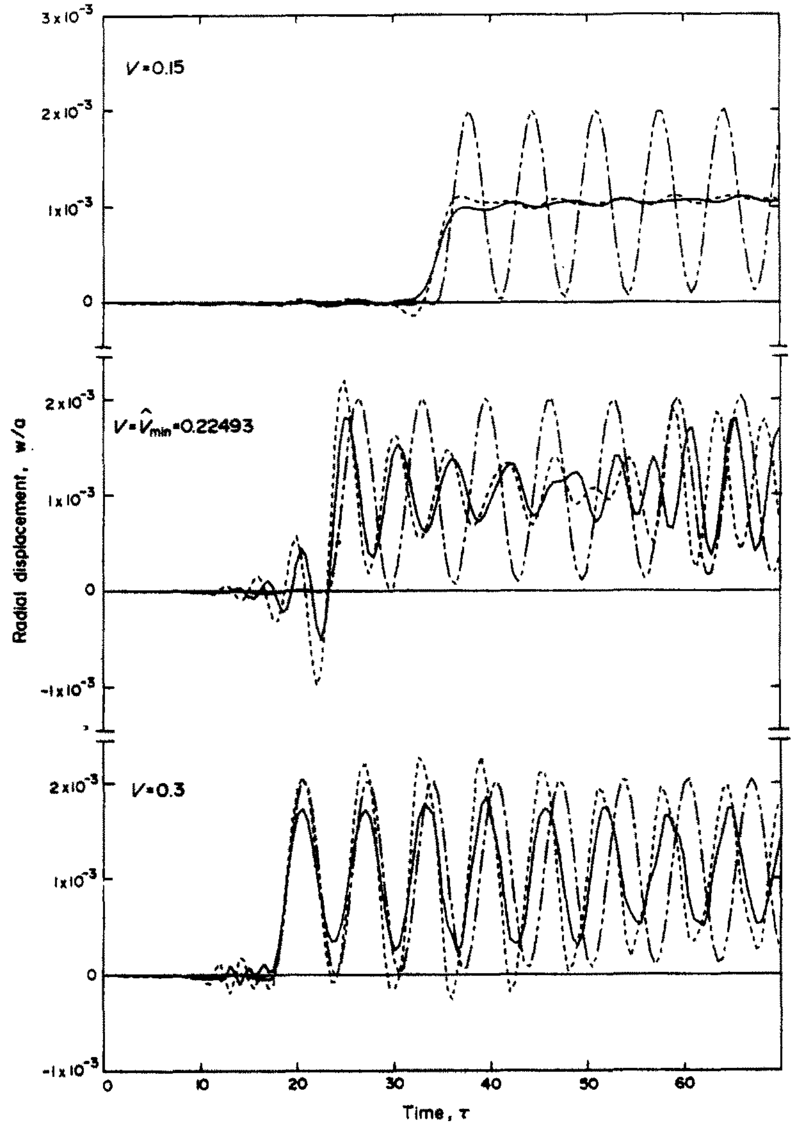

Fig. 2. Dimensionless radial displacement w/a vs dimensionless time, at station $\xi = 5.2$, for pressure front speeds $V < \hat{V}_r$. *Notes*: (——), With shear deformation ($\kappa = 5/6$) and rotatory inertia; (– – – – –), Love theory; (— - — -), Membrane theory; $\nu = 0.3$, $L/a = 10$, $h/a = 0.1$, $E/E_0 = 0.7$, $c_p t_1/a = 100$, $q_0 a(1-\nu^2)/(E_0 h) = -0.00091$.

shell model because at these speeds the transient response of simply-supported, semi-infinite, elastic shells becomes unbounded as shown in [10]. The simpler shell models have dispersion curves with only two branches. The upper branches are in excellent agreement with the most exact model; the lower branches agree at long wavelengths but differ greatly from each other and the most exact model in the short wavelength region.

The responses in Figs. 2 and 3 were calculated by truncating the eigenfunction series (56) for all three models at 100 terms when $V \neq \hat{V}_{min}$ and 200 terms when $V = \hat{V}_{min}$. The adequacy of this representation can be deduced from the dispersion curves as follows. Corresponding to each eigenvalue Ω_n, one can find from (74) a pair of values which will be called $\hat{V}_n$ and $\hat{f}_n$. If these values are plotted for the first twenty eigenvalues on top of the dispersion curves, say for the most exact shell model, one gets the points shown in Fig. 4. It can be demonstrated that the most significant contribution in the eigenfunction series comes from the low frequency terms for which the ratio $V/\hat{V}_n$ is nearest to 1. It is clear from Fig. 4 that when the eigenfunction series is truncated beyond 20 terms, the dominant terms in the series are accounted for, at least in the cases of the most exact shell model and Love's model. Beyond the dominant terms, the absolute values of the Fourier coefficients in the displacement series solutions for these two

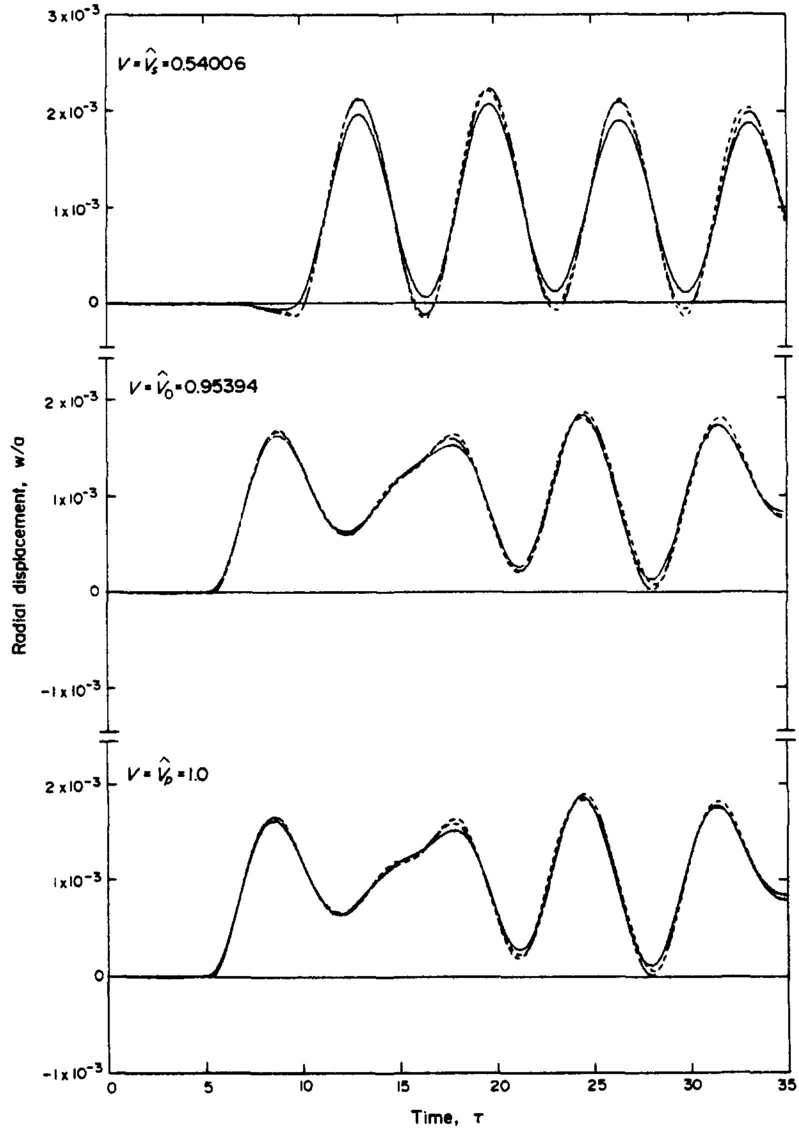

Fig. 3. Dimensionless radial displacement w/a vs dimensionless time at station $\xi = 5.2$, for pressure front speeds $V > \hat{V}_s$. *Notes*: (——), With shear deformation ($\kappa = 5/6$) and rotatory inertia; (-------), Love theory; (— — —), Membrane theory; $\nu = 0.3$, $L/a = 10$, $h/a = 0.1$, $E_\infty/E_0 = 0.7$, $c_p t_1/a = 100$, $q_0 a(1-\nu^2)/(E_0 h) = -0.00091$.

models go to zero at least as the inverse of the square of the harmonic numbers; thus, the series converge absolutely and uniformly. By summing the stated number of terms, an accuracy of better than five decimals was achieved.

The membrane model requires special consideration because the lower branch of its dispersion curves approaches asymptotically the zero phase velocity line as the wave number goes to infinity. This property manifests itself also in the arrangement of the eigenvalues Ω_n by causing an infinite number of them to form, starting with the lowest one, a strictly increasing sequence that converges to the finite limit point $\sqrt{(1-\nu^2)}$. As a result, the eigenfunction series as constructed in (56) does not converge to the solution of the membrane problem because the series excludes the eigenfunctions whose eigenvalues belong to the upper branch of the dispersion diagram; instead, these terms must be added separately. In the calculations, the total number of eigenvalues considered was arbitrarily split equally between the two branches. The absolute values of the Fourier coefficients associated with the eigenvalues belonging to the upper dispersion curve go to zero at least as the inverse of the square of the harmonic numbers; those associated with the lower branch go at least as the inverse. By summing the stated number of terms in the displacement series, an accuracy of only five decimals was achieved.

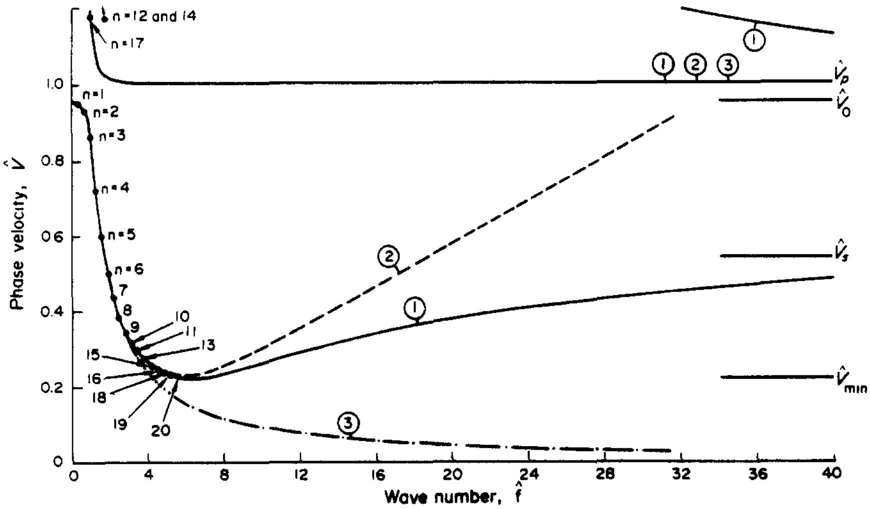

Fig. 4. Dispersion curves for elastic shell: phase velocity vs wave number. *Notes*: Harmonic wave, phase $= \hat{f}(\hat{V}\tau - \xi)$, $h/a = 0.1$, $\nu = 0.3$, ①, With shear deformation ($\kappa = 5/6$) and rotatory inertia; ②, Love theory; ③, Membrane theory; ·, Points $(\hat{V}_n, \hat{f}_n)$.

From Figs. 2 and 3, it is seen that all three shell models predict small radial motions before the arrival of the pressure front at $\xi = 5.2$. Following passage of the front, the three models predict oscillatory radial motion at load speeds $V > \hat{V}_{min}$. At speeds $V < \hat{V}_{min}$, the most exact and Love models predict a nearly static response while the membrane model predicts large oscillations. Until disturbances initiated from the far end of the shell arrive at $\xi = 5.2$, the oscillations can be characterized at load speeds V other than $\hat{V}_{min}$ as steady-state harmonic waves propagating with phase velocity V and wavelength $2\pi/\hat{f}_n$ corresponding to $V/\hat{V}_n = 1$. The foregoing is true also at $V = \hat{V}_{min}$ but only for the membrane shell. The most exact and Love models by contrast predict oscillations at this speed that decay immediately behind the pressure front. This results from the fact that the radial disturbances generated have a group velocity nearly equal to $\hat{V}_{min}$ and so do not spread as rapidly behind the pressure front as at other load speeds. Since the static responses show little creep and since the largest oscillation amplitudes do not decrease appreciably with time, the response can be said to be nearly elastic at all load speeds. At load speeds $\hat{V}_0$ and $\hat{V}_{min}$, critical for semi-infinite elastic shells, no extraordinarily large radial displacements occur.

From the above results and from the relation of the dominant term in the series solutions to the dispersion curves, it follows that at load speeds $V \geq \hat{V}_s$, the responses of all three models are virtually the same. Thus, at these load speeds, the moving load problem can, for all practical purposes, be studied with the membrane theory alone. At $\hat{V}_{min} < V < \hat{V}_s$, there are significant differences between the responses of all three models and, therefore, at these speeds, it is necessary to account for shear deformation and rotatory inertia. At speeds $V < \hat{V}_{min}$, the Love and most exact models agree, and hence, at such low load speeds the Love model is sufficient for analyzing the moving load problem. It should perhaps be remarked that the foregoing observations were for a nearly elastic shell. It was found that the acceptability of the membrane model improves at load speeds $V < \hat{V}_s$ with increasing material dissipation.

Acknowledgement—The contents of this paper constituted a part of the first author's Ph.D. dissertation at the University of California, Berkeley, where he held a Hughes Doctoral Fellowship.

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